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THE STABILITY OF SYSTEMS WITH ELASTIC COMPONENTS AND SLIDER OSCILLATIONS†

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The problem of the asymptotic stability of a linear system, whose parameters belong to a certain set, is considered. Stability criteria for a number of typical special cases are obtained. The results should find application when investigating the dynamics of mechanical systems with constraints, as shown by the example of the oscillations of a slider on a rough plane. © 2006 Elsevier Ltd. All rights reserved.

Similar problems were solved when analysing the effect of the force structure on the stability of equilibrium. In particular, the Thomson–Tait–Chetayev theorems [1] hold for general assumptions regarding the dissipative forces. Some results on stability under parametric uncertainty were presented in [2].

1. FORMULATION OF THE PROBLEM AND GENERAL RESULTS

We will investigate the stability of the following system

$$A\ddot{q} + B\dot{q} + Cq = 0, \quad q \in R^n, \quad A, B, C \in R^{n^2} \tag{1.1}$$

assuming that the non-degenerate constant matrix A is specified and, as regards the constant matrices B and C, we only know that they belong to a certain class of matrices K.

Problems of the dynamics of systems of rigid bodies in which the method of deformable elements is used to determine the reactions lead to a similar formulation. If q is the strain vector, the matrices B and C describe the dissipation and elastic forces. The form of these matrices is determined by expressions for the potential energy of deformations and the Rayleigh dissipative function, which are unknown a priori. The only thing that can be said with certainty is that these matrices are symmetrical and positive. The matrix A connects the reactions with the generalized accelerations, and it is traditionally associated with the mass distribution. In the case of non-ideal constraints the friction law is also taken into account in this matrix, and it can be asymmetrical.

Stability under uncertainty is called robust stability [2]. General results on robust stability have so far only been obtained for certain special cases, for which the range of variation of the parameters is limited. In this paper we consider unbounded matrix classes. It is obvious that the wider the class K the stricter the requirements imposed on the matrix A in order to ensure stability for any $B, C \in K$.

We will first consider a well-known special case.

Theorem 1. Suppose $K = S_+$ is a class of symmetrical positive-definite matrices. For asymptotic stability of system (1.1) for any $B, C \in K$ it is necessary and sufficient that $A \in K$.

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Proof. The sufficiency follows from the third Thomson–Tait–Chetayev theorem [1]. To prove the necessity we carry out an orthogonal transformation of the coordinates in system (1.1), as a result of which the symmetrical part of the matrix A takes the diagonal form $||a_{ii}||$ $(i = 1, \ldots, n)$. Assuming the matrices B and C to be diagonal, we set up the characteristic equation

$$\det\|A\lambda^2 + B\lambda + C\| = 0 \tag{1.2}$$

In Eq. (1.2) the coefficient of λ^2 is a polynomial of the first degree in a_{ii} with positive coefficients, which depend on the elements of the matrices B and C. The positiveness of this polynomial, necessary for stability, means that $a_{ii} \geq 0$ $(i = 1, \ldots, n)$. We will further assume that there is a non-zero element outside the principal diagonal in matrix A, for example $a_{12} \neq 0$. Then, the second-order corner minor of matrix (1.2) is a fourth-degree polynomial, the coefficients of which depend on the elements b_{ii} and c_{ii} $(i = 1, 2)$ of the matrices B and C. A check of the Hurwitz conditions shows that if $a_{11}/a_{22} = b_{11}/b_{22} = c_{11}/c_{22}$, where $a_{12}^2 c_{11} > a_{11} b_{11}^2$, the polynomial has a root in the right half-plane. Taking the large parameters into account, we obtain that the characteristic polynomial (1.2) also has this root. The contradiction obtained shows that $a_{12} = 0$. Finally, from the condition for the matrix A to be degenerate we obtain that $a_{ii} > 0$ $(i = 1, \ldots, n)$, whence it follows that $A \in K$.

Another special case is the class of positive scalar matrices. System (1.1) takes the form

$$A\ddot{q} + b\dot{q} + cq = 0 \tag{1.3}$$

where b and c are positive numbers.

Theorem 2. For the asymptotic stability of system (1.3) for any b and c, it is necessary and sufficient that all the eigenvalues of the matrix A should be positive real numbers.

Proof. By means of a non-degenerate linear transformation of system (1.3) we reduce the matrix A to real normal form (in this case only the first term in Eq. (1.3) is changed). The following second-order equation corresponds to every real eigenvalue λ_a of this matrix in characteristic equation (1.2)

$$\lambda_a \lambda^2 + b\lambda + c = 0$$

the roots of which lie in the left half-plane only when $\lambda_a > 0$. The following fourth-order equation corresponds to complex eigenvalues $\alpha \pm i\beta$

$$\begin{vmatrix} \alpha\lambda^2 + b\lambda + c & \beta\lambda^2 \\ -\beta\lambda^2 & \alpha\lambda^2 + b\lambda + c \end{vmatrix} = p_4\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0$$

$$p_4 = \alpha^2 + \beta^2, \quad p_3 = 2\alpha b, \quad p_2 = b^2 + 2\alpha c, \quad p_1 = 2bc, \quad p_0 = c^2 \tag{1.4}$$

The Hurwitz conditions for a fourth-order polynomial are

$$p_j > 0, \quad p_1 p_2 p_3 > p_0 p_3^2 + p_1^2 p_4, \quad j = 0, 1, \ldots, 4 \tag{1.5}$$

Substituting the values (1.4) into conditions (1.5), we arrive at the single inequality

$$\alpha b^2 > c\beta^2$$

which is satisfied for any positive b and c only when $\alpha > 0$ and $\beta = 0$, which has also been confirmed.

The classes of matrices considered in Theorems 1 and 2 correspond to two cases: in Theorem 1 no restrictions are imposed on the deformed components, while in Theorem 2 a model of similar linear viscoelastic components is used. We will discuss some intermediate cases below.

2. THE CASE OF TWO INDEPENDENT NON-LINEAR COMPONENTS

Suppose $n = 2$, and, as the class K, we consider the class D of diagonal matrices with positive elements. From the practical point of view this case corresponds to the model of two independent deformable components, no necessarily the same or linear.

Theorem 3. For the asymptotic stability of system (1.1), where $n = 2$, for any $B, C \in D$ it is necessary and sufficient for the following conditions to be satisfied

$$a_{11} > 0, \quad a_{22} > 0, \quad \Delta = \det A > 0, \quad a_{12}a_{21} > 0 \tag{2.1}$$

Proof. Expanding determinant (1.2), we obtain the following expressions for the coefficients of characteristic equation (1.4)

$$\begin{aligned} p_4 &= \Delta, \quad p_3 = a_{11}b_{22} + a_{22}b_{11}, \quad p_2 = a_{11}c_{22} + a_{22}c_{11} + b_{11}b_{22} \\ p_1 &= b_{11}c_{22} + b_{22}c_{11}, \quad p_0 = c_{11}c_{22} \end{aligned} \tag{2.2}$$

We will first assume that one of the inequalities (2.1) has the opposite meaning. If the first inequality is violated, then as $b_{11} \to 0$ we will have $p_3 < 0$. If $\Delta < 0$ then $p_4 < 0$. Finally suppose $a_{12}a_{21} < 0$. Substituting expressions (2.2) into the last condition of (1.5) we arrive at the inequality

$$\begin{aligned} &-a_{12}a_{21}(b_{11}c_{22} + b_{22}c_{11})^2 < b_{11}b_{22}(a_{11}b_{22} + a_{22}b_{11})(b_{11}c_{22} + b_{22}c_{11}) + \\ &+ b_{11}b_{22}(a_{11}c_{22} - a_{22}c_{11})^2 \end{aligned} \tag{2.3}$$

The last term on the right-hand side of this inequality can be made zero by choosing c_{11} and c_{12}. Then, the left-hand side will have a quadratic form in b_{11} and b_{12}, while the right-hand side will have a fourth-order form in these coefficients. Hence, when $b_{11}, b_{22} \to 0$, inequality (2.3) is not satisfied, which indicates instability.

The sufficiency of conditions (2.1) following from the fact that the left-hand side of inequality (2.3) is negative, while the right-hand side is positive.

Suppose D_2 is a set of second-order diagonal matrices with positive elements, for which the second element is not less than the first.

From the practical point of view, the use of the set D_2 as the class K indicates that the model of two similar non-linear deformable components is chosen, where, for these conditions, the second component is loaded more than the first.

Theorem 4. For the asymptotic stability of system (1.1), where $n = 2$, for any $B, C \in D_2$ it is necessary and sufficient for the following conditions to be satisfied

$$\begin{aligned} &a_{11} > 0, \quad a_{11} + a_{22} > 0, \quad \Delta > 0, \quad a_{12}a_{21} > \Phi(a_{11} - a_{22}) \\ &\Phi(x) = \begin{cases} 0, & \text{if} \quad x \le 0 \\ -x^2/4, & \text{if} \quad x > 0 \end{cases} \end{aligned} \tag{2.4}$$

Proof. The first two inequalities of (2.4) are equivalent to the requirement $p_3 > 0$ for all $b_{22} \ge b_{11} > 0$. To clarify the meaning of the last inequality of (2.4) we divide both sides of condition (2.3) by $b_{11}b_{22}$. We finally obtain

$$\begin{aligned} &-a_{12}a_{21}(c_{11}\gamma + \gamma^{-1}c_{22})^2 < (a_{11}b_{22} + a_{22}b_{11})(b_{11}c_{22} + b_{22}c_{11}) + (a_{11}c_{22} - a_{22}c_{11})^2 \\ &\gamma = \sqrt{b_{22}/b_{11}} \ge 1 \end{aligned} \tag{2.5}$$

The first term on the right-hand side of this inequality is positive, and it can be made as small as desired without changing the remaining terms of this formula. If $a_{12}a_{21} > 0$, inequality (2.5) is satisfied, and if $a_{12}a_{21} < 0$, its left-hand side reaches a maximum when $\gamma = 1$.

Consequently, condition (2.5) is equivalent to the following inequality

$$-a_{12}a_{21} < (a_{11}\delta - a_{22})^2/(1+\delta)^2; \quad \delta = c_{22}/c_{11} \ge 1 \tag{2.6}$$

If $a_{11} \le a_{22}$, the minimum value of the right-hand side of inequality (2.6) is equal to zero. When $a_{11} > a_{22}$, the minimum value is reached when $\delta = 1$. Both these possibilities are combined in the last inequality of (2.4).

Remarks. 1. We will define D_1 as a set of second-order diagonal matrices with positive elements, for which the first element is not less than the second. Then, an assertion similar to Theorem 4 will hold, but with the coefficients a_{11} and a_{22} interchanged.

2. The non-satisfaction of conditions (2.1) or (2.4) indicates that for certain $B, C \in K$ the system is unstable. Moreover, we can state the conditions of instability for any $B, C \in K$. When $K = D$ these include the cases when the first two inequalities of (2.1) or the third inequality have opposite meanings [3]. If $K = D_2$, violation of the third inequality of (2.1) (when $p_4 < 0$) or of the first two inequalities of (2.4) (when $p_3 < 0$) leads to instability.

3. If neither the conditions of Theorems 3 or 4 nor the conditions in Remark 2 are satisfied, then, depending on the choice of the matrices $B, C \in K$, we can have both stability and instability. The transition from stability to instability when there is a continuous change in the elements of these matrices occurs in accordance with the Hopf bifurcation scenario, since the characteristic equation in all the cases considered has no zero roots.

3. OSCILLATIONS OF A SLIDER ON A ROUGH PLANE

We will consider a rigid body in the form of parallelepiped, which slides on a rough horizontal plane. This system is already well known from Coulomb's works on determining the laws of friction. Nevertheless, certain aspects of the dynamics of a slider remain unclear. These include the oscillations of a body in a vertical direction, observed in practice. For the case of a position, which moves between two guides, these oscillations have been explained [4] by resonance between the natural frequencies of the system. Possible reasons for these are also [5] collisions between microroughnesses and oscillations of the supports (microseisms). To explain the phenomenon of the "squealing" of brake shoes, a model of four linear deformable components with specified characteristics was used in [6]. Below we propose a model of two similar non-linear components, for which the stiffness and viscosity coefficients increase together with the deformations.

We will assume that the motion occurs in a vertical plane, which is the plane of symmetry of the body. We will direct the coordinate axes horizontally in the sliding direction and vertically upwards. The fundamental theorems of dynamics are expressed by the equations

$$m\ddot{x} = X - \mu N, \quad m\ddot{y} = Y + N, \quad mk^2\ddot{\varphi} = M + M_N - \mu bN \tag{3.1}$$

where m is the mass of the body, a and b are the half-lengths of its edges, k is the radius of inertia, x, y and φ are the coordinates of the centre of mass and the angle of rotation of the body, μ is the friction coefficient, X, Y and M are the external forces and their moments, and N and M_N is the principal vector and principal moment of the normal reaction.

We will mentally place the deformable components at the corner points of the body, in which case

$$N = N_1 + N_2, \quad M_N = a(N_2 - N_1) \tag{3.2}$$

We have the following equations for the deformations of the components

$$\delta_1 = a\varphi - y, \quad \delta_2 = -a\varphi - y \tag{3.3}$$

Substituting expressions (3.2) and (3.3) into Eqs (3.1), we obtain

$$mk^2\ddot{\delta}_i = (-1)^{i+1}[aM + a^2(N_2 - N_1) - \mu ab(N_1 + N_2)] - k^2(Y + N_1 + N_2), \quad i = 1, 2 \tag{3.4}$$

Replacing the left-hand side in Eq. (3.4) by zero, we can obtain the equilibrium values of the reactions, and then the equilibrium deformations δ_1^0, δ_2^0. Then, linearizing system (3.4) in the neighbourhood of equilibrium, we obtain a system of the form (1.1) in the perturbations $q_i = \delta_i - \delta_i^0$, where

$$b_i = \frac{\partial N_i(\delta_i^0, 0)}{\partial \dot{q}_i}, \quad c_i = \frac{\partial N_i(\delta_i^0, 0)}{\partial q_i}, \quad A = \frac{mk^2}{4a^2}\|a_{ij}\|; \quad i, j = 1, 2$$

$$a_{11} = 1 + U - V, \quad a_{12} = -1 + U - V, \quad a_{21} = -1 + U + V, \quad a_{22} = 1 + U + V$$

$$U = a^2/k^2, \quad V = \mu ab/k^2$$

We will consider the problem of the sliding stability of a slider with respect to the variables δ_i, $\dot{\delta}_i$ ($i = 1, 2$). Since these variables define the motion of the centre of mass of the body along the vertical, and also its rotational motion, the presence of asymptotic stability indicates that the oscillations along the vertical, due to the initial perturbations, decay with time. We will use the results obtained above to investigate the stability.

It can be verified that the conditions of Theorem 2 are satisfied for all $U, V > 0$. This means that the model of linear components is unsuitable for describing non-decaying oscillations.

The conditions of Theorem 3 reduce to the single inequality

$$V < |U - 1|$$

which indicates that oscillations do not occur if the friction coefficient does not exceed a certain threshold

$$\mu < \mu^* = |a^2 - k^2|/(ab) \qquad (3.5)$$

For a homogeneous slider $k^2 = (a^2 + b^2)/3$, whence

$$\mu^* = |2a^2 - b^2|/(3ab) \qquad (3.6)$$

It can be seen that the critical value of the friction coefficient depends on the shape of the body and, in principle, can be as close to zero as desired. A body in the shape of a standard sheet of paper, standing on the short side, possesses the greatest instability ($\mu^* = 0$).

Theorem 4, where $K = D_2$, must be used when the equilibrium solutions of system (3.4) satisfy the condition $N_2 > N_1$, which is satisfied, in particular, if the moment of the external forces M is equal to zero. Since $a_{22} > a_{11}$, the stability condition also has the form (3.5).

We will assume, finally, that the external moment ensures that the inequality $N_2 < N_1$ is satisfied. Then, the stability conditions are satisfied for all values of $U, V > 0$.

It should be noted that for the values

$$\mu > \mu^{**} = |a^2 + k^2|/(ab)$$

we obtain $a_{11} < 0$, which indicates that equilibrium values of the reaction, determined from system (3.4), do not exist or are non-unique [3].

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